RIGID PUNCH INDENTATION INTO A PLASTIC HALF-SPACE

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The indentation of punches of wedge-like and rectangular planform into a plastic half-space is investigated in this paper.

These problems are further developments of known results obtained by Prandtl [1]. It should be noted that the corresponding axisymmetrical problems have been solved by Ishlinskii [13] and Shield [9].

First, the theory of the state of a spherical deformation, of which the theory of the state of a plane deformation is a special case, is considered. The generalization is based on properties of various coordinate systems, to include some other systems as special cases. Thus, the rectangular cartesian coordinate system can be looked at as a degenerate case of a cylindrical and spherical systems, etc.

In the case analyzed in this paper the spherical system constitutes the original.

1. The equilibrium equation in spherical coordinates has the following form.

$$\begin{aligned}
&\frac{\partial\sigma_\rho}{\partial_\rho} + \frac{1}{\rho}\frac{\partial\tau_{\rho\theta}}{\partial\theta} + \frac{1}{\rho\sin\theta}\frac{\partial\tau_{\rho\varphi}}{\partial\varphi} + \frac{1}{\rho}[2\sigma_\rho - \sigma_\theta - \sigma_\varphi + \tau_{\rho\theta}\operatorname{ctg}\theta] = 0\\
&\frac{\partial\tau_{\rho\theta}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{\rho\sin\theta}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi} + \frac{1}{\rho}[(\sigma_\theta - \sigma_\varphi)\operatorname{ctg}\theta + 3\tau_{\rho\theta}] = 0 \qquad (1.1)\\
&\frac{\partial\tau_{\rho\varphi}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\tau_{\theta\varphi}}{\partial\theta} + \frac{1}{\rho\sin\theta}\frac{\partial\sigma_\varphi}{\partial\varphi} + \frac{1}{\rho}[3\tau_{\rho\varphi} + 2\tau_{\theta\varphi}\operatorname{ctg}\theta] = 0
\end{aligned}$$

The angle θ is measured between the radii and the positive direction of the z-axis, ϕ is the angle measured around the z-axis to the right.

Denoting by u, v, w the displacement velocities along the axes, the expressions for the components of the rate of deformation tensor are

$$\varepsilon_\rho = \frac{\partial u}{\partial \rho}, \qquad \varepsilon_{\theta\varphi} = \frac{1}{2\rho\sin\theta}\left(\sin\theta\frac{\partial w}{\partial\theta} - w\cos\theta + \frac{\partial v}{\partial\varphi}\right)$$

$$\varepsilon_\theta = \frac{1}{\rho}\left(\frac{dv}{d\theta} + u\right), \qquad \varepsilon_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{\rho\sin\theta}\frac{\partial u}{\partial\varphi} + \frac{\partial w}{\partial\rho} - \frac{w}{\rho}\right)$$

$$\varepsilon_\varphi = \frac{1}{\rho\sin\theta}\left(\frac{\partial w}{\partial\varphi} + u\sin\theta + v\cos\theta\right), \quad \varepsilon_{\rho\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial\varphi} - \frac{v}{\rho} + \frac{1}{\rho}\frac{\partial u}{\partial\theta}\right) \tag{1.2}$$

Let us assume that

$$u = \rho u^*(\theta, \varphi), \qquad v = \rho v^*(\theta, \varphi), \qquad w = \rho w^*(\theta, \varphi) \tag{1.3}$$

Obviously, in this case the components ϵ_{ij} are independent of the quantity ρ. It then follows that it is possible to seek a solution in the form $\sigma_{ij} = \sigma_{ij}(\theta, \phi)$.

Let us further assume that

$$\tau_{\rho\theta} = \tau_{\rho\varphi} = 0 \tag{1.4}$$

Then from the first equilibrium equation we get

$$\sigma_\rho = \frac{1}{2}(\sigma_\theta + \sigma_\varphi) \tag{1.5}$$

It should be noted that relationship (1.5) follows from (1.4) and the equilibrium equations.

We also assume that the Tresca plasticity condition is fulfilled. From (1.4) and (1.5) it follows that the state of stress cannot correspond to the edge of the Tresca prism (the condition of full plasticity). It corresponds to the face of this prism. Because of (1.4) and (1.5) the third invariant of the stress deviation tensor is zero. Thus, following [8], we get

$$(\sigma_\theta - \sigma_\varphi)^2 + 4\tau_{\theta\varphi}^2 = 4k^2 \tag{1.6}$$

$$\varepsilon_\rho = u^* = 0, \qquad \varepsilon_\theta = \frac{\partial v^*}{\partial\theta} = \lambda(\sigma_\theta - \sigma_\varphi)$$

$$\varepsilon_\varphi = \frac{1}{\sin\theta}\left(\frac{\partial w^*}{\partial\varphi} + v^*\cos\theta\right) = \lambda(\sigma_\varphi - \sigma_\theta) \tag{1.7}$$

$$\varepsilon_{\theta\varphi} = \frac{1}{2\sin\theta}\left(\sin\theta\frac{\partial w^*}{\partial\theta} - w^*\cos\theta + \frac{\partial v^*}{\partial\varphi}\right) = 2\lambda\tau_{\theta\varphi}, \quad \varepsilon_{\rho\theta} = \varepsilon_{\rho\varphi} = 0$$

It is easy to see that under conditions (1.3) and (1.4), relationships (1.6) and (1.7) will also be satisfied for the von Mises plasticity condition (except for a constant term on the right-hand side of (1.6). The equations (1.1) will now have the following form:

$$\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+(\sigma_\theta-\sigma_\varphi)\operatorname{ctg}\theta=0 \qquad (1.8)$$
$$\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\sigma_\varphi}{\partial\varphi}+2\tau_{\theta\varphi}\operatorname{ctg}\theta=0$$

The resulting state is called a spherical deformation state, analogously to the plane deformation state. Indeed, if the latter is considered to take place on some plane, the former takes place on some spherical surface. Obviously the spherical deformation state also includes the plane deformation state as a special case. To show this it is sufficient to perform the following change of coordinates:

$$x=R\varphi,\qquad y=R(\theta-{}^1/_2\pi)\qquad(\varphi\to 0,\ \theta\to{}^1/_2\pi,\ R\to\infty)$$
$$u_x=Rw^*,\qquad u_y=Rv^*\quad(v^*\to 0\ \ w^*\to 0)$$

where u_x, u_y are the displacements along the x- and y-axes, respectively.

The relationship (1.6) is satisfied by setting

$$\sigma_\theta=2kp+k\cos 2\psi,\quad \sigma_\varphi=2kp-k\cos 2\psi,\quad \tau_{\theta\varphi}=k\sin 2\psi \qquad (1.9)$$

Substituting (1.9) into (1.8) we get

$$\frac{\partial p}{\partial\theta}-\sin 2\psi\frac{\partial\psi}{\partial\theta}+\frac{\cos 2\psi}{\sin\theta}\frac{\partial\psi}{\partial\varphi}+\cos 2\psi\operatorname{ctg}\theta=0$$
$$\cos 2\psi\frac{\partial\psi}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial p}{\partial\varphi}+\frac{\sin 2\psi}{\sin\theta}\frac{\partial\psi}{\partial\varphi}+\sin 2\psi\operatorname{ctg}\theta=0 \qquad (1.10)$$

The characteristics of equation (1.10) are

$$\left(\frac{d\varphi}{d\theta}\right)_{1,2}=\frac{\operatorname{tg}(\psi\pm{}^1/_4\pi)}{\sin\theta} \qquad (1.11)$$

It is easy to prove that along the line (1.11) the following relationships are satisfied:

$$dp\pm d\psi\pm\cos\theta d\varphi=0 \qquad (1.12)$$

These relationships are generalizations of well known Hencky integrals [3].

Now consider equations for the displacement velocities. The relationships (1.7) yield two equations for the determination of the two components v^* and w^*. One of these equations is

$$\varepsilon_\theta+\varepsilon_\varphi=0$$

and the second can be recorded in the following form

$$\frac{\varepsilon_{\theta\varphi}}{\varepsilon_\theta-\varepsilon_\varphi}=\frac{\tau_{\theta\varphi}}{\sigma_\theta-\sigma_\varphi} \tag{1.13}$$

Rewriting these equations in terms of displacement velocity components, taking into account (1.9), and dropping the asterisks, we get

$$\frac{\partial v}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}+v\operatorname{ctg}\theta=0$$
$$2\frac{\partial v}{\partial\theta}\sin 2\psi-\cos 2\psi\left(\frac{\partial w}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}-w\operatorname{ctg}\theta\right)=0 \tag{1.14}$$

It is easy to show that the characteristics of (1.14) can be expressed by (1.11). Along these characteristics the relations which are the generalizations of the known Hencky relationships [4] will be satisfied.

$$\cos\theta d\varphi\,[v\operatorname{tg}(\psi\pm{}^1/_2\pi)-w]+dv+dw\operatorname{tg}(\psi\pm{}^1/_4\pi)=0 \tag{1.15}$$

(This is seen if we consider (1.15) as a relationship between the velocity components along the characteristics.)

In order to investigate the properties of the characteristics, we utilize the method of investigation proposed by Khristianovich [5],[6].

Let us assume that $\alpha(\theta,\ \phi)$ = const and $\beta(\theta,\ \phi)$ = const are the equations of the two families of characteristics.

Then

$$\frac{\partial\varphi}{\partial\alpha}-\frac{\operatorname{tg}(\psi+{}^1/_4\pi)}{\sin\theta}\frac{\partial\theta}{\partial\alpha}=0,\qquad \frac{\partial p}{\partial\alpha}+\frac{\partial\psi}{\partial\alpha}+\cos\theta\frac{\partial\varphi}{\partial\alpha}=0 \tag{1.16}$$
$$\frac{\partial v}{\partial\alpha}+\operatorname{tg}\left(\psi+\frac{\pi}{4}\right)\frac{\partial w}{\partial\alpha}+\cos\theta\left[v\operatorname{tg}\left(\psi+\frac{\pi}{4}\right)-w\right]\frac{\partial\varphi}{\partial\alpha}=0$$

as well as

$$\frac{\partial\varphi}{\partial\beta}-\frac{\operatorname{tg}(\psi-{}^1/_4\pi)}{\sin\theta}\frac{\partial\theta}{\partial\beta}=0,\qquad \frac{\partial p}{\partial\beta}-\frac{\partial\psi}{\partial\beta}-\cos\theta\frac{\partial\varphi}{\partial\beta}=0$$
$$\frac{\partial v}{\partial\beta}-\operatorname{tg}\left(\psi-\frac{\pi}{2}\right)\frac{\partial w}{\partial\beta}+\cos\theta\left[v\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)-w\right]\frac{\partial\varphi}{\partial\beta}=0 \tag{1.17}$$

Assuming that $\psi=\psi\ (\alpha)$, $v=v\ (\alpha)$, $w=w\ (\alpha)$, it will be possible to consider α as a parameter of ψ. Hence, equations (1.7) will acquire the form

$$\varphi-\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)\ln\operatorname{tg}\frac{\theta}{2}=\Phi_1(\psi),\quad p-\psi+\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)\ln\sin\theta=\Phi_2(\psi) \tag{1.18}$$
$$v-\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)w+\left[v\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)-w\right]\operatorname{tg}\left(\psi-\frac{\pi}{4}\right)\ln\sin\theta=\Phi_3(\psi)$$

The first family of the characteristics consists of the lines (1.18) for $\psi =$ const. The second family is found by integrating the systems (1.16),

$$\varphi - \int \frac{\operatorname{tg}(\psi + {}^1/_4\pi)}{\sin\theta} d\theta = \Phi_4(\beta)$$

$$p + \psi + \int \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) \operatorname{ctg}\theta d\theta = \Phi_5(\beta) \tag{1.19}$$

$$v + \int \operatorname{tg}\left(\psi + \frac{\pi}{2}\right) dw + \int \left[v \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) - w\right] \cos\theta d\varphi = \Phi_6(\beta)$$

The case $\psi = \psi\ (\beta)$ is considered in an analogous manner. Let $\psi \equiv$ const. In this case

$$\varphi - \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) \ln \operatorname{tg}\frac{\theta}{2} = \Theta_1(\beta) \tag{1.20}$$

$$p + \psi + \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) \ln \sin\theta = \Theta_2(\beta)$$

$$v + \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) w + \int \left[v \operatorname{tg}\left(\psi + \frac{\pi}{4}\right) - w\right] \cos\theta d\varphi = \Theta_3(\beta)$$

$$\varphi + \operatorname{tg}\left(\psi - \frac{\pi}{4}\right) \ln \operatorname{tg}\frac{\theta}{2} = \Theta_4(\alpha)$$

$$p - \psi - \operatorname{tg}\left(\psi - \frac{\pi}{4}\right) \ln \sin\theta = \Theta_5(\alpha)$$

$$v - \operatorname{tg}\left(\psi - \frac{\pi}{4}\right) w + \int \left[v \operatorname{tg}\left(\psi - \frac{\pi}{4}\right) - w\right] \cos\theta d\varphi = \Theta_6(\alpha)$$

2. The results of the theory of plane deformation can be generalized for the case of spherical deformation.

Consider a generalization of Prandtl's solution [1] of the problem of a rigid punch on a plastic half-space. Let $\theta =$ const; this respresents a part of a half-space bounded by a circular cone. The generalized Prandtl punch in this case will be in contact with the half-space over a part of a surface bounded by two radii.

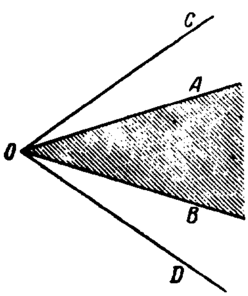

Fig. 1.

Of the greatest interest is the simplest case, when $\theta = 1/2\ \pi$. In this case we have the indentation into a plastic half-space by a wedge-shaped punch. Figure 1 represents the projection of the punch on the boundary plane. Note that an analogous elastic problem has been considered by Galin [7].

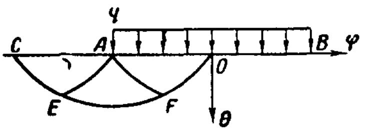

Fig. 2.

Consider an auxiliary $\theta\phi$ plane (Fig. 2), for $-\pi \leqslant \phi \leqslant \pi$, $\theta \geqslant 1/2\,\pi$. Assume that normal pressure is acting along the line AB which produces a plastic state in the region CAOFE, and similarly in the symmetrical region.

The stress distribution in the regions $CA\ E$ and AOE is clearly independent of ϕ. From (1.10) we therefore obtain

$$\sin 2\psi = \frac{c_1}{\sin^2\theta}, \quad c_1 = \text{const} \tag{2.1}$$

Since, however, $\tau_{\theta\phi} = 0$ for $\theta = 1/2\,\pi$, then $c_1 = 0$ in (2.1), and, consequently, $\tau_{\theta\phi} = 0$ everywhere in the regions ACE and AOF.

Next we find that in these regions

$$p = \varkappa \ln \sin\theta + c_2, \qquad c_2 = \text{const} \tag{2.2}$$

where $\kappa = -1$ for $\psi = 0, \pm\pi, \ldots,$; $\kappa = 1$ for $\psi = \pm\pi/2, \pm 3/2\,\pi$. Assuming that in the regions CAE the stress component σ_ϕ is a compressive stress, we can put $\psi = 0$. Thus, from the condition $\sigma_\theta = 0$ for $\theta = 1/2\,\pi$, for this region we get

$$c_2 = -{}^1/_2$$

Assuming that in the region AOF the component σ_ϕ is a tensile stress, and that the pressure on AO equals q, and putting $\psi = 1/2\,\pi$, for this region we get

$$c_2 = \frac{1}{2}(q + 2k)/k$$

The unknown pressure q is obtained by matching the regions CAE and AOF by the construction of a plastic region EAF. The latter is obviously a generalization of the fan-region introduced by Prandtl.

The integral should be taken

$$p - \psi = \operatorname{tg}(\psi - {}^1/_2\pi) \ln \sin\theta + c_3, \qquad c_3 = \text{const}$$

From the matching condition we obtain

$$q = -k(2+\pi)$$

Thus, the pressure necessary to produce plastic deformation by a wedge-shaped punch on a half-space is shown to be identical with the pressure to produce the same effect by a punch in the shape of an infinite strip (Prandtl's solution).

The displacement velocity field is determined from (1.14). Now consider the plastic flow along the rigid boundary $OFEC$, Figure 2. On the boundary AO the normal velocity is given, $v = 1$. Since in the regions CAE and AOF $\sin 2\psi = 0$, the flow in these regions is shearless. On the boundaries AF and AE there are possible velocity discontinuities along the tangential directions to these boundaries.

The problem of the determination of the displacement velocity field can be solved numerically. It will give the change of the coordinate net in some instant close to the initial instant of indentation. Clearly, the maximum value of the angle AOB (Fig. 1) in the case analyzed here is $2/3\,\pi$.

3. Applying Prandtl's method [2], we obtain another exact solution of the theory of ideal plasticity.

Assume that $\tau_{\theta\phi} = f(\theta)$. From (1.6), (1.8) we get

$$\sigma_\theta - \sigma_\varphi = 2\mu\sqrt{k^2 - f^2(\theta)}, \qquad \mu = \operatorname{sign}(\sigma_\theta - \sigma_\varphi) \tag{3.1}$$

$$\frac{\partial\sigma_\theta}{\partial\theta} + (\sigma_\theta - \sigma_\varphi)\operatorname{ctg}\theta = 0, \qquad \sin\theta\frac{df}{d\theta} + \frac{\partial\sigma_\varphi}{\partial\varphi} + 2f\operatorname{ctg}\theta = 0 \tag{3.2}$$

Putting

$$\sin\theta\frac{df}{d\theta} + 2f\operatorname{ctg}\theta = B_1 \qquad (B_1 = \text{const})$$

we find

$$f(\theta) = \frac{B_2 - B_1\cos\theta}{\sin^2\theta} \qquad (B_2 = \text{const}) \tag{3.3}$$

From (3.1), (3.2), (3.3) we find that

$$\sigma_\theta = -2\mu\int\sqrt{k^2 - f^2(\theta)}\operatorname{ctg}\theta\, d\theta + \chi(\varphi)$$

$$\sigma_\varphi = -2\mu\int\sqrt{k^2 - f^2(\theta)}\operatorname{ctg}\theta\, d\theta - 2\mu\sqrt{k^2 - f^2(\theta)}\operatorname{ctg}\theta + \chi(\varphi)$$

It is obvious that

$$\chi(\varphi) = B_1\varphi$$

Putting $v = v\ (\theta)$, $w = w\ (\theta)$, we determine the displacement velocity field as follows:

$$v = \frac{B_3}{\sin\theta},$$

$$w = \left[2\mu \int \frac{f(\theta)\cos\theta}{\sqrt{k^2 - f^2(\theta)}} \frac{d\theta}{\sin^3\theta} + B_4\right]\sin\theta$$

Where B_3 and B_4 are some constants.

4. Assume that the punch pressing on the half-space has a rectangular projection, shown in Figure 3 as a rectangle $ABCO$.

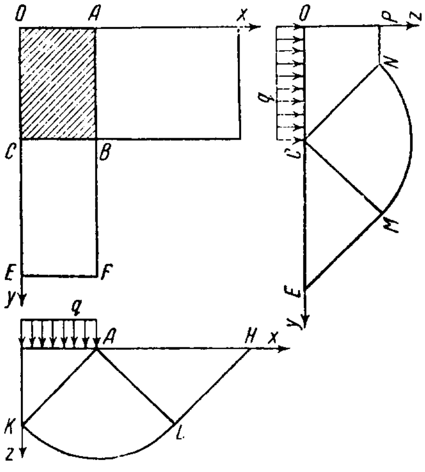

Fig. 3.

Assume also that the plasticity condition is as expressed in [8].

$$\begin{aligned}
\tau_{xy}^2 &= (\sigma_x - \sigma \pm \tfrac{2}{3}k)(\sigma_y - \sigma \pm \tfrac{2}{3}k) \\
\tau_{yz}^2 &= (\sigma_y - \sigma \pm \tfrac{2}{3}k)(\sigma_z - \sigma \pm \tfrac{2}{3}k) \qquad (4.1) \\
\tau_{zx}^2 &= (\sigma_z - \sigma \pm \tfrac{2}{3}k)(\sigma_x - \sigma \pm \tfrac{2}{3}k)
\end{aligned}$$

where

$$\sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z), \qquad k = \text{const}$$

The plasticized portions of the material appear on the surface of the half-space in the regions $BCEF$ and $ABCH$ Figure 3.

We will assume that in the region $ABCO$ a uniformly distributed pressure q is acting.

Obviously, the equilibrium conditions and the plasticity conditions (4.1) are satisfied for

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0, \qquad \sigma_z = q, \qquad \sigma_x = \sigma_y = q + 2k \tag{4.2}$$

It is also obvious that in the plasticized regions $BCEF$ and $ABCH$ we must put

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0, \qquad \sigma_z = 0, \qquad \sigma_x = \sigma_y = -2k \tag{4.3}$$

The unknown pressure q is determined from the matching conditions of the plastic regions under the areas $ABCO$ and $BCEF$, $ABCH$, Figure 3. For this purpose we utilize the regions of the type shown in Figure 4, which we will call sector regions. To find the stress and strain distribution in these regions we use cylindrical coordinates $\rho\ \theta\ \xi$. Assuming that all stress components in the sector depend only on the angle θ, we get

$$\frac{d\tau_{\rho\theta}}{d\theta} + \sigma_\rho - \sigma_\theta = 0, \qquad \frac{d\sigma_\theta}{d\theta} + 2\tau_{\rho\theta} = 0, \qquad \frac{d\tau_{\theta\xi}}{d\theta} + \tau_{\rho\xi} = 0 \tag{4.4}$$

Next, assuming $\tau_{\theta\xi} = \tau_{\rho\xi} = 0$, we obtain

$$\tau_{\rho\theta} = k, \qquad \sigma_\rho = \sigma_\theta = 2k(c - \theta) \qquad (0 \leqslant \theta \leqslant {}^1/_2\pi)$$
$$\sigma_\xi = 2k(c-\theta) - k, \qquad c = \mathrm{const} \tag{4.5}$$

In the regions which lie at the angle $1/4\ \pi$ to the axis, where the conditions (4.2) and (4.3) are fulfilled, we have

$$\tau_n = k, \qquad \sigma_n = q + k, \qquad \tau_n = -k, \qquad \sigma_n = -k \tag{4.6}$$

Thus, from the matching conditions it is easy to find that

$$q = -k(2 + \pi) \tag{4.7}$$

On the matching boundaries of the plastic regions all stress components are continuous except for σ_ξ on AK and CN, Figure 3.

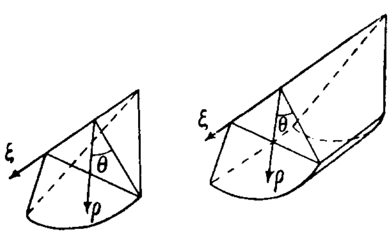

Fig. 4.

The value of the discontinuity of the modulus is $2k$. The discontinuity of the σ_ξ stress is statically admissible.

We note that Shied and Drucker [11] have showed that the pressure satisfies the inequality

$$5k \leqslant |q| \leqslant 5.71k$$

We now proceed to construct the velocity field. For this purpose we use the following relationships, [10],

$$\varepsilon_x + \varepsilon_{xy} \frac{\sigma_y - \sigma \pm {}^2/_3 k}{\tau_{xy}} + \varepsilon_{xz} \frac{\sigma_z - \sigma \pm {}^2/_3 k}{\tau_{xz}} = \varepsilon_{xy} \frac{\sigma_x - \sigma \pm {}^2/_3 k}{\tau_{xy}} +$$

$$+ \varepsilon_y + \varepsilon_{yz} \frac{\sigma_z - \sigma \pm {}^2/_3 k}{\tau_{yz}} = \varepsilon_{xz} \frac{\sigma_x - \sigma \pm {}^2/_3 k}{\tau_{xz}} + \varepsilon_{yz} \frac{\sigma_y - \sigma \pm {}^2/_3 k}{\tau_{yz}} + \varepsilon_z \qquad (4.8)$$

where

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

Following Prandtl, we assume that the plasticized region of the material under the punch is moving with a unit velocity downward.

$$u_x = u_y = 0, \quad u_z = 1$$

In the sector regions

$$u_\rho = u_\xi = 0, \qquad u_0 = \frac{\sqrt{2}}{2} \qquad (4.9)$$

It is clear that the velocity field (4.9) satisfies relationships (3.8). In the plasticized regions under the areas $BCEF$ and $ABGH$ we put

$$u_y = \frac{\sqrt{2}}{2}, \quad u_z = \frac{\sqrt{2}}{2}; \qquad u_x = \frac{\sqrt{2}}{2}, \quad u_z = -\frac{\sqrt{2}}{2}$$

It is not difficult to verify that the solution can be now constructed following the work of Hill [12]. The stress distribution is determined in an analogous manner, as has been done above. The construction of the velocity field, however, requires some small modifications.

In summary then, it has been shown that Prandtl's formula (4.7) is also valid for rectangular punches acting on a half-space.

A further extension of the results obtained would be the solution of problems of polygonal punch. The solution of these problems, however, would require numerical computations, since the magnitude of the pressure in this case is not constant.

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